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ERRATUM

Absence of the absolutely continuous spectrum for Stark–Bloch operators with strongly singular periodic potentials

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Abstract. We correct here the proof of the boundedness of the coupling term $\mathbf{X}$ given by us in a previous paper (1995 *J. Phys. A: Math. Gen.* **28** 1101–6).

In [MS] we stated the absence of the absolutely continuous spectrum for Stark–Bloch self-adjoint operators formally defined on $L^2(\mathbb{R}, \mathrm{d}x)$ as

$$H_f = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{j\in\mathbb{Z}} \alpha\delta'(x-j) + fx \qquad f>0,\ \alpha\neq 0.$$

See [AEL] for the physical interest of this problem and see [E] where a proof of the absence of the absolutely continuous spectrum is given. Recently, [ADE] have proved that the spectrum is purely point for some values of f and α strong enough.

The proof we gave in [MS] contains a technical mistake in lemma 3. Here we correct the proof of lemma 3 [MS] which is based on the incorrect claim $\|\mathbf{X}\|^2 = \max_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} |X_{n,m}|^2$, just after equation (13) of [MS], where $\|\cdot\|$ denotes the norm

$$\|\mathbf{X}\| := \sup_{\xi\in\ell^2(\mathbb{N}),\xi\neq 0} \frac{\|\eta\|_{\ell^2(\mathbb{N})}}{\|\xi\|_{\ell^2(\mathbb{N})}}$$

where $\xi = (\xi_1,\ldots,\xi_n,\ldots) \in \ell^2(\mathbb{N})$ and $\eta = \mathbf{X}\xi = (\eta_1,\ldots,\eta_m,\ldots) \in \ell^2(\mathbb{N})$ is given by $\eta_m = \sum_{n=1}^{\infty} X_{m,n}\xi_n$. In the following [$n$, MS] denotes the formula (n) of the paper [MS].

The elements $X_{n,m}$ are functions, depending on a real parameter k belonging to the torus $\mathcal{B} = \mathbb{R}/2\pi\mathbb{Z} = [-\pi,\pi)$, given by [12, MS].

We have the following theorem.

Theorem. For any $k\in\mathcal{B}$ the operator $\mathbf{X}(k)$, represented by the matrix $\{X_{n,m}(k)\}_{n,m\in\mathbb{N}}$, is a bounded operator from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$. That is there exists a positive constant C such that for any $\xi\in\ell^2(\mathbb{N})$

$$\|\mathbf{X}(k)\xi\|_{\ell^2(\mathbb{N})} \leqslant C\|\xi\|_{\ell^2(\mathbb{N})}.$$

The constant C is independent of k.

Proof. As a first step we improve the estimate [13, MS] (we refer to [MS] for the notation).

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Lemma. We have that $X_{n,m}(k) = x_{n,m} + R^1_{n,m}(k) + R^2_{n,m}(k)$ where

$$x_{n,m} = \begin{cases} \frac{1}{2} & n = m \\ 0 & n+m \text{ even, } n \neq m \\ -\frac{2}{\pi}\left[\frac{\sin((n+m)\pi/2)}{n+m} + \frac{\sin((n-m)\pi/2)}{n-m}\right] & n+m \text{ odd} \end{cases}$$

and

$$R^1_{n,m}(k) = \mathcal{O}((n-m)^{-2}) \qquad R^2_{n,m}(k) = \mathcal{O}(q^{-1}(n-m)^{-1})$$

as $n - m$ and q go to infinity, $q = \min(n, m)$, and where the asymptotic behaviour is uniform for $k \in \mathcal{B}$.

Proof. Let

$$u_n(x,k) = C_n(k)v_n(x,k)$$

where $C_n := C_n(k) = 1/\sqrt{2} + \mathcal{O}(n^{-1})$ is a normalization constant (see [9, MS] and [10, MS], in the following, for the sake of simplicity, we drop k if not necessary) and (see [7, MS])

$$v_n(x,k) = \begin{cases} e^{-ikx}[e^{iK_n x} - e^{-iK_n x}\, e^{-iK_n}\tilde{w}_n(k)] & x \in \left(-\frac{1}{2}, 0\right) \\ e^{-i(kx+k+K_n)}[e^{iK_n x} - e^{-iK_n x}\, e^{iK_n}\tilde{w}_n(k)] & x \in \left(0, \frac{1}{2}\right) \end{cases}$$

where

$$K_n := K_n(k) = n\pi + \frac{2}{n\pi}(1 - (-1)^n \cos k) + \mathcal{O}(n^{-2}) \tag{1}$$

solves the equation

$$\cos(k) = \cos(K_n) - \tfrac{1}{2}K_n \sin(K_n) \tag{2}$$

and

$$\begin{aligned}\tilde{w}_n(k) &= \frac{e^{ik} - e^{-iK_n}}{1 - e^{i(k-K_n)}} = \frac{\cos(k) - \cos(K_n)}{1 - \cos(k - K_n)} = \frac{\cos(k) - \cos(K_n)}{1 - \cos(k)\cos(K_n) - \sin(k)\sin(K_n)} \\ &= \frac{-K_n \sin(K_n)/2}{1 - \cos^2(K_n) + K_n \sin(K_n)\cos(K_n)/2 - \sin(k)\sin(K_n)} \\ &= \frac{-1}{\cos(K_n) + 2[\sin(K_n) - \sin(k)]/K_n} = [-(-1)^n + \mathcal{O}(n^{-1})]\end{aligned}$$

since (1) and (2). From this it follows that

$$v_n(x,k) = \begin{cases} e^{-ikx}[2\cos(K_n x) - e^{-iK_n x} r^-_n(k)] & x \in \left(-\frac{1}{2}, 0\right) \\ e^{-i(kx+k+K_n)}[2\cos(K_n x) - e^{-iK_n x} r^+_n(k)] & x \in \left(0, \frac{1}{2}\right) \end{cases}$$

where

$$r^\pm_n(k) = (1 + e^{\pm iK_n}\tilde{w}_n(k)) = \mathcal{O}(n^{-1}). \tag{3}$$

Let us stress that the same behaviour also holds for the derivative of $r^\pm_n(k)$ (denoted by $'$) because (see [MS])

$$K'_n(k) = \frac{2(-1)^n \sin(k)}{n\pi}[1 + \mathcal{O}(n^{-1})]. \tag{4}$$

Now, we are ready to calculate the coupling term $X_{n,m}(k)$ for $n \neq m$:

$$X_{n,m}(k) = \mathrm{i} \int_{-1/2}^{1/2} \bar{u}_n(x,k) \frac{\partial u_m(x,k)}{\partial k}\, \mathrm{d}x = C_m(k)\bar{C}_n(k) Y_{n,m}(k)$$

where

$$Y_{n,m}(k) = \mathrm{i} \int_{-1/2}^{1/2} \bar{v}_n(x,k) \frac{\partial v_m(x,k)}{\partial k}\, \mathrm{d}x \tag{5}$$

because

$$\bar{C}_n(k) C_m(k) \int_{-1/2}^{1/2} \bar{v}_n(x,k) v_m(x,k)\, \mathrm{d}x = \int_{-1/2}^{1/2} \bar{u}_n(x,k) u_m(x,k)\, \mathrm{d}x = \delta_n^m.$$

From this it follows also that $X_{n,m}(k) = -\bar{X}_{m,n}(k)$. The integral (5) takes the form $Y_{n,m} = Y_{n,m}^1 + Y_{n,m}^2$ where

$$\begin{aligned} Y_{n,m}^1(k) &= \mathrm{i} \int_{-1/2}^{0} \bar{v}_n(x,k) \frac{\partial v_m(x,k)}{\partial k}\, \mathrm{d}x \\ &= 4 \int_{-1/2}^{0} x \cos(K_n x) \cos(K_m x)\, \mathrm{d}x + \mathcal{O}(m^{-1}(m-n)^{-1}) \end{aligned}$$

because (1), (3) and (4), and

$$\begin{aligned} Y_{n,m}^2(k) &= \mathrm{i} \int_{0}^{1/2} \bar{v}_n(x,k) \frac{\partial v_m(x,k)}{\partial k}\, \mathrm{d}x \\ &= 4s \int_{0}^{1/2} (x+1) \cos(K_n x) \cos(K_m x)\, \mathrm{d}x + \mathcal{O}(m^{-1}(m-n)^{-1}) \end{aligned}$$

where we set

$$s = \mathrm{e}^{\mathrm{i}(K_n - K_m)} = (-1)^{n-m}[1 + \mathcal{O}((n-m)^{-1})]. \tag{6}$$

Then we have that

$$Y_{n,m}(k) = Z_{n,m}(k) + \mathcal{O}(m^{-1}(m-n)^{-1})$$

where

$$\begin{aligned} Z_{n,m}(k) &= 4 \int_{0}^{1/2} [s + x(s-1)] \cos(K_n x) \cos(K_m x)\, \mathrm{d}x \\ &= 2[s + (s-1)/2] \left[\frac{\sin[(K_n + K_m)/2]}{K_n + K_m} + \frac{\sin[(K_n - K_m)/2]}{K_n - K_m} \right] \\ &\quad + 2(s-1) \left[\frac{\cos[(K_n + K_m)/2] - 1}{(K_n + K_m)^2} + \frac{\cos[(K_n - K_m)/2] - 1}{(K_n - K_m)^2} \right]. \end{aligned}$$

From this and from (1) and (6) it follows that $Z_{n,m}(k) = \mathcal{O}((m-n)^{-2})$ when $n+m$ is even and

$$Z_{n,m}(k) = -4 \left[\frac{\sin[(n+m)\pi/2]}{(n+m)\pi} + \frac{\sin[(n-m)\pi/2]}{(n-m)\pi} \right] + \mathcal{O}((m-n)^{-2}) \tag{7}$$

when $n+m$ is odd. Because $C_n(k)\bar{C}_m(k) = \frac{1}{2}[1 + \mathcal{O}(q^{-1})]$, $q = \min(n,m)$, then the statement is proved for $n \neq m$. For $n = m$ we have

$$X_{n,n}(k) = \mathrm{i}|C_n(k)|^2 \int_{-1/2}^{1/2} \bar{v}_n(x,k) \frac{\partial v_n(x,k)}{\partial k}\, \mathrm{d}x + \mathrm{i}\bar{C}_n(k) C_n(k)' \int_{-1/2}^{1/2} |v_n(x,k)|^2\, \mathrm{d}x$$

where $C_n(k) = 1/\sqrt{2} + \mathcal{O}(n^{-1})$ and $C_n(k)' = \mathcal{O}(n^{-1})$ (see [9, MS] and [10, MS]) and

$$\mathrm{i}\int_{-1/2}^{1/2} \bar{v}_n(x,k)\frac{\partial v_n(x,k)}{\partial k}\,\mathrm{d}x = 1 + \mathcal{O}(n^{-1})$$

by using similar arguments used to estimate $Y_{n,m}$. Hence

$$X_{n,n}(k) = \tfrac{1}{2} + \mathcal{O}(n^{-1}).$$

□

From the lemma we can write

$$\mathbf{X} = \mathbf{D} + \mathbf{H}^{+} + \mathbf{H}^{-} + \mathbf{T} + \mathbf{R}^{1} + \mathbf{R}^{2} \tag{8}$$

where $\mathbf{D} = \mathrm{diag}(X_{n,n}(k))$,

$$\mathbf{H}^{+} = \{H^{+}_{n,m}\}_{n,m\in\mathbb{N}} \qquad H^{+}_{n,m} = c^{+}_{n+m} \qquad c^{+}_{j} = \begin{cases} 2/j\pi & j = 4\ell+3, \ell = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{H}^{1} = \{H^{-}_{n,m}\}_{n,m\in\mathbb{N}} \qquad H^{+}_{n,m} = c^{-}_{n+m} \qquad c^{-}_{j} = \begin{cases} -2/j\pi & j = 4\ell+1, \ell = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{T} = \{T_{n,m}\}_{n,m\in\mathbb{N}} \qquad T_{n,m} = t_{n-m} \qquad t_{j} = \begin{cases} -(-1)^{\ell}2/j\pi & j = 2\ell+1, \ell \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

and $\mathbf{R}^{j} = \{R^{j}_{n,m}\}_{n,m\in\mathbb{N}}$, $j = 1, 2$, are remainder terms such that $R^{1}_{n,m}(k) = \mathcal{O}((m-n)^{-2})$ and $R^{2}_{n,m}(k) = \mathcal{O}(q^{-1}(m-n)^{-1})$ for any $k \in [-\pi, \pi]$.

As we shall show later we have that the matrices $\mathbf{D}$, $\mathbf{H}^{+}$, $\mathbf{H}^{-}$, $\mathbf{T}$, $\mathbf{R}^{1}$ and $\mathbf{R}^{2}$ represent bounded operators from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$ for any $k \in [-\pi, \pi]$; so the sum (8) is well defined and it represents a bounded operator from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$.

Because

$$\sup_{n}\sum_{m\in\mathbb{N}}|R^{1}_{n,m}| < \infty \qquad \text{and} \qquad \sup_{m}\sum_{n\in\mathbb{N}}|R^{1}_{n,m}| < \infty \tag{9}$$

then it follows that $\mathbf{R}^{1}$ defines a bounded operator from $\ell^1(\mathbb{N})$ to $\ell^1(\mathbb{N})$ (see theorem 2.13 in [M2]) as well as from $\ell^\infty(\mathbb{N})$ to $\ell^\infty(\mathbb{N})$ (see theorem 2.6 in [M2]). Hence, it defines a bounded operator from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$ (see theorem 9, section 7.1 in [M1]). Because

$$\sum_{n,m=1}^{\infty}|R^{2}_{n,m}(k)|^2 \leqslant c\sum_{n=1}^{\infty}\left[\sum_{m=1}^{n-1}\frac{1}{m^2(n-m)^2} + \sum_{m=n+1}^{\infty}\frac{1}{(n+1)^2(n-m)^2}\right]$$

for some positive constant c, where

$$\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{1}{(n+1)^2(n-m)^2} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(n+1)^2m^2} < \infty$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{n-1}\frac{1}{m^2(n-m)^2} = \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}\frac{1}{m^2(n-m)^2} < \infty$$

then it follows that $\mathbf{R}^{2}$ defines a bounded operator from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$; this is an immediate consequence of the Schwarz inequality (see, e.g. remark 36, p 21, in [H]). The diagonal matrix $\mathbf{D}$ represents a bounded operator from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$ because it differs from $\frac{1}{2}1$, where 1 is the unit matrix, up to a diagonal matrix having terms satisfying (9).

The two matrices $\mathbf{H}^+$ and $\mathbf{H}^-$ are of Hilbert type and so they define bounded operators from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$ (see problem 38, p 23 of [H]).

The matrix $\mathbf{T}$ is a Toeplitz matrix given by

$$\mathbf{T} = \begin{pmatrix} 0 & -2/\pi & 0 & 2/3\pi & 0 & -2/5\pi & \dots \\ -2/\pi & 0 & -2/\pi & 0 & 2/3\pi & 0 & \dots \\ 0 & -2/\pi & 0 & -2/\pi & 0 & 2/3\pi & \dots \\ 2/3\pi & 0 & -2/\pi & 0 & -2/\pi & 0 & \dots \\ 0 & 2/3\pi & 0 & -2/\pi & 0 & -2/\pi & \dots \\ -2/5\pi & 0 & 2/3\pi & 0 & -2/\pi & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Toeplitz matrix $\mathbf{T} = \{t_{n-m}\}_{n,m\in\mathbb{N}}$ is associated to the function

$$\Phi(z) = \sum_{j\in\mathbb{Z}} t_j z^j \qquad z \in C,\ |z| = 1$$

and the operator represented by $\mathbf{T}$ and acting from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$ is bounded if the function $\Phi(z)$ is essentially bounded on the complex circle of radius 1; in particular, we have that (see corollary 3.2, Ch XXIII in [GGK])

$$\|\mathbf{T}\| = \operatorname{ess}\sup_{\theta\in[0,2\pi)} |\Phi(\mathrm{e}^{\mathrm{i}\theta})|.$$

In our case we have that

$$\Phi(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{j\in\mathbb{Z}} t_j\, \mathrm{e}^{\mathrm{i}\theta j} = \sum_{j=1}^{\infty} t_j(\mathrm{e}^{\mathrm{i}\theta j} + \mathrm{e}^{-\mathrm{i}\theta j}) = -\frac{4}{\pi}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2\ell+1}\cos[(2\ell+1)\theta] = -p_{\pi/2}(\theta)$$

where $p_{\pi/2}(\theta)$ is the periodic step function which is equal to 1 for $|\theta| < \pi/2$, it is equal to 0 for $\theta = \pm\pi/2$ and it is equal to -1 for $|\theta| \in (\pi/2, \pi]$. Hence, $\mathbf{T}$ is a bounded operator.

Therefore, we can conclude that $\mathbf{X}$ is a bounded operator for any k and the constant C in the statement is independent of k because all the asymptotic behaviours are uniform with respect to $k \in \mathcal{B}$. □

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